*Proof.* From Theorem 2 with $\alpha=\beta=1/2$ we obtain

$$\frac{1}{2}\left[a-1+\frac{1}{a}\right]+\frac{1}{2}\left[b-1+\frac{1}{b}\right]\geqq 1,$$

$$\frac{1}{2}\,[a+b]+\frac{1}{2}\left[\frac{1}{a}+\frac{1}{b}\right]\geqq 2,$$

from which the required inequality follows.

COROLLARY 2.2. *If* $\lambda>0$ *then* $\lambda+(1/\lambda)\geqq 2$.

*Proof.* Let $\lambda=a=b>0$ in Corollary 2.1.

## THE CATENARY AND THE TRACTRIX

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Although the facts herein have been in the public domain for a long time, our purpose is to re-establish them in an elementary way—then set them on parade in a relatively complete and ordered line of march. Note that we carefully avoid rectangular equations (*per se*) which, at least for the tractrix, have an awesome appearance. Moreover, once a start is made, facts seem to tumble over each other in their demand for recognition.

**1. The catenary.** A flexible and inextensible chain weighing $m$ pounds per foot hangs from two supports $A$ and $B$. Equilibrium throughout is established if the three forces: $T$, the magnitude of the variable tangential force at a repre-

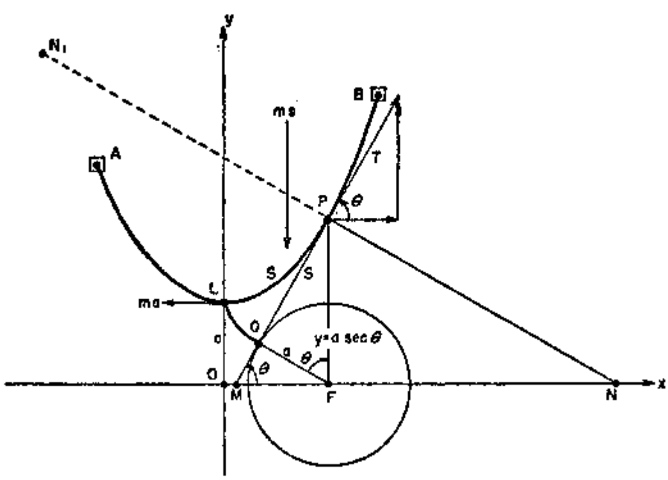


FIG. 1

sentative point $P$; $ma$, the constant* horizontal force at the lowest point $L$; and $ms$, the vertical weight of the chain length $s=\widehat{LP}$; have a resultant of zero magnitude. That is, if $\theta$ is the inclination of $T$:

$$T \sin \theta = ms, \qquad T \cos \theta = ma, \tag{1}$$

so that

$$s = a \tan \theta, \tag{2}$$

an *intrinsic* equation of the curve, called the catenary. Since the chain hangs steady, $T$ is tangent to its locus at $P$ and $\tan \theta$ is the slope of the curve.

In Figure 1, horizontal and vertical $x$, $y$ axes are drawn with origin $O$ distant $a$ units below $L$. Let $F$ be the foot of the ordinate of $P$ and draw $FQ$ perpendicular to the tangent line $PT$. Then, since $\sphericalangle QFP=\theta$,

$$PQ = ks, \qquad FQ = ka, \qquad y = ka\cdot \sec \theta.$$

But for $\theta=0$, $y=a$, and thus $k=1$. Accordingly, the tangent to the catenary at any point is also tangent to the circle with center $F$ and radius $a$.

From Equation (1), the tension is

$$T = ma\cdot \sec \theta = my, \tag{3}$$

a quantity equal to the weight of a length of the chain hanging vertically from $P$ to $F$.

Since $y=a\cdot\sec\theta$ and generally $dx=(\cos\theta)ds$, then $ydx=(a\sec\theta)(\cos\theta)ds$ $=a\cdot ds$ and thus

$$\text{Area } (OLPF) = \int_0^x ydx = \int_0^s ads = as.$$

That is,

$$\text{Area } (OLPF) = 2\cdot \text{Area } (\Delta FQP). \tag{4}$$

Furthermore, since $ydx=ads$ or $\pi y^2dx=\pi ayds$, volume $V_x$ and surface area $\sum_x$ of revolution about the $x$-axis have the special relation

$$2V_x = a \textstyle\sum_x. \tag{5}$$

Let $PN$ be the normal length from $P$ to the $x$-axis. Then, if $R$ is the radius of curvature at $P$, we have by definition from Equation (2):

$$R = \left|\frac{ds}{d\theta}\right| = a \sec^2 \theta = \frac{y^2}{a} = PN. \tag{6}$$

The center of curvature, however, is at $N_1$ opposite $N$ from $P$.

* This is evident if we imagine resupporting the chain with pegs at various points $P$. The shape of the chain does not change and thus the tension at $L$ is constant in direction and magnitude.

**2. The tractrix.** In Figure 1, $PQ = s = \widehat{PL}$ and thus the locus of $Q$ is an *involute* of the catenary, a curve called the tractrix. Its tangent length $QF$ is con-

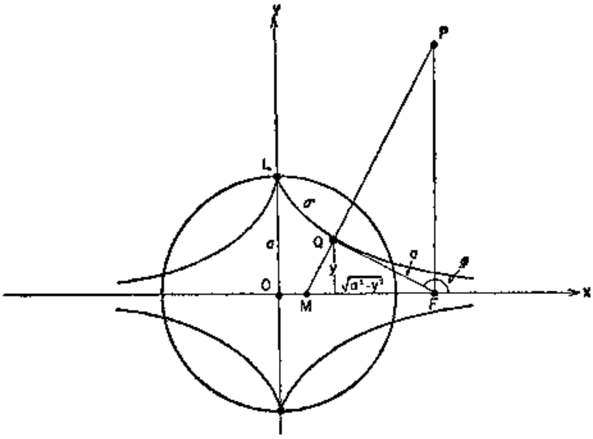


FIG. 2

stant and equal to $a$, and the curve can be pictured as the path of a toy wagon $Q$ pulled along by a child $F$. It is quite obviously the orthogonal trajectory of circles of fixed radius $a$ having centers on a line.

If $\phi$ is the inclination of the tangent, then the expression

$$\tan\phi = y' = \frac{y}{\pm\sqrt{a^2 - y^2}}, \qquad y = a,\ x = 0 \tag{7}$$

defines four branches as shown in Figure 2.

Interesting and useful properties of the tractrix are now established directly from this differential equation (7) which we write as $ydx = \pm\sqrt{a^2 - y^2}dy$. Note first, however, that if $\sigma = \widehat{LQ}$, $dy = (\sin\phi)d\sigma$ and, particularly here, $y = a\sin\phi$. The following measures of area $A$, radius of curvature $\rho$, volume $V_x$ and surface area $\sum_x$ of revolution about the $x$-axis are immediate.

$$A = 2\int_{-\infty}^{\infty} ydx = 2\int_{-a}^{a}\sqrt{a^2 - y^2}\,dy = \pi a^2;^*$$

$$V_x = \pi\int_{-\infty}^{\infty} y^2dx = 2\pi\int_0^a \sqrt{a^2 - y^2}\,(ydy) = \frac{2}{3}\pi a^3;$$

$$\sum\nolimits_x = 2\pi\int_{-\infty}^{\infty} yd\sigma = 2\pi\int_0^a (a\sin\phi)(\csc\phi\ dy) = 2\pi\int_0^a ady = 2\pi a^2;$$

$$\rho = \frac{d\sigma}{d\phi} = |\,a\cot\phi\,| = |\,-a\tan\theta\,| = PQ.$$

* The form $\int_{-a}^{a}\sqrt{a^2 - y^2}dy$ measures the half-area of the circle $x^2 + y^2 = a^2$ shown in Figure 2.

The last item is recognized as the involute-evolute relation of the catenary and tractrix. From it,

$$\sigma = |a \ln \sin \phi|, \tag{8}$$

an intrinsic equation of the tractrix.

The tractrix and its surface of revolution, called the *pseudosphere*, thus have much in common with the circle and sphere. This striking analogy appears even stronger if curvatures of the two surfaces are compared.

This curvature is determined as follows. A plane containing the normal to a surface at $Q$ intersects the surface in a curve of curvature $K$. As the plane turns about the fixed normal, $K$ may attain maximum and minimum values $K_1$ and $K_2$. Their *product* $K_1K_2$ is defined as the curvature of the surface at $Q$. These values $K_1$, $K_2$ occur in sections at right angles to each other.*

For a sphere all plane sections through a normal are great circles of radius $a$ and curvature $1/a$. A sphere then has curvature $1/a^2$.

For the pseudosphere, the section of minimum curvature at $Q$ is made by the plane through the axis of revolution; the maximum radius of curvature is $QP$. The minimum radius of curvature at $Q$ is $QM$, formed by the plane perpendicular to the first. Their product $(QM)(QP) = -(FQ)^2 = -a^2$ is constant and negative since the radii are oppositely directed. Thus $K_1K_2 = -1/a^2$.

The plane, the sphere, and the pseudosphere are surfaces upon which we may display the parabolic geometry of Euclid, the elliptic geometry of Riemann, and the hyperbolic geometry of Lobatschewsky and Bolyai. These are characterized by the angle-sum $A+B+C$ of triangles formed by geodesics (lines of shortest distance):

$$A + B + C \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 180^\circ \begin{cases} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic.} \end{cases}$$

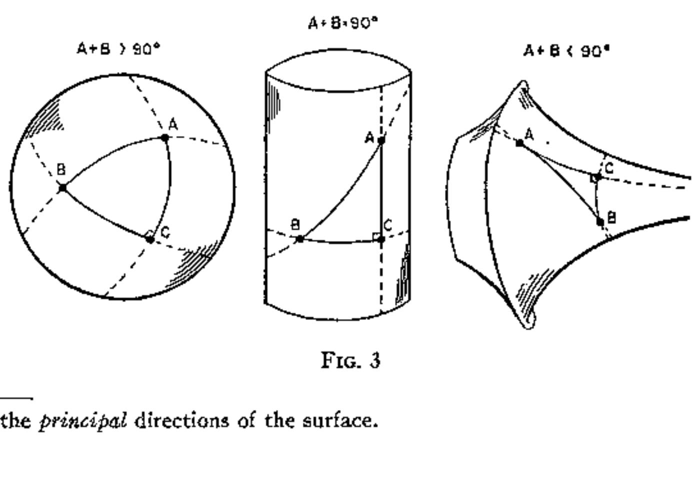


FIG. 3

* Along the *principal* directions of the surface.

A right circular cylinder may replace the plane. Figure 3 gives these surfaces showing triangles with $C=90°$. Their sides are each geodesics from point to point.

On the cylinder $A$ and $C$ are taken on an element, $B$ and $C$ on a circle. $A$ and $B$ lie on a *helix*.

On the sphere the sides are great circles.

On the pseudosphere $A$ and $C$ are on a tractrix (a meridian of the surface), $B$ and $C$ on a circle.

It appears that upon the sphere $A+B>90°$ and upon the pseudosphere $A+B<90°$. The fact that $A+B=90°$ on the cylinder is evident if we imagine the cylinder as a roller in a printing press. The image of $ABC$ printed on plane paper is a triangle with straight sides. Moreover, on the cylinder

$$\widehat{AB}^2 = \widehat{BC}^2 + \overline{CA}^2.$$

As a final item, consider a pivot seated in a step. As the pivot turns, most wear occurs on the surface farthest from the axis of rotation. In time the seat

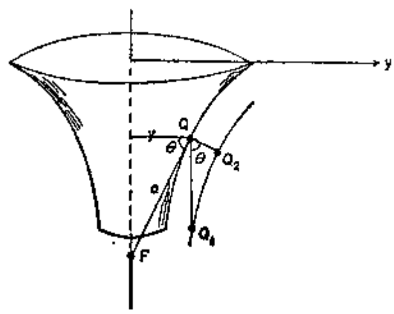


FIG. 4

and pivot become incompatible and wobble occurs. We seek the shape of a pivot such that the wear $QQ_1$, Figure 4, parallel to the axis of rotation is the same for all points $Q$ of the pivot. Thus, as action and wear go on, the pivot will reseat itself.

The amount of wear $QQ_2$ normal to the section curve at $Q(x, y)$ is proportional to the work done by friction as the pivot turns. Let $f$ be the coefficient of friction, $p$ the pressure of the bearing, and $n$ the number of revolutions per unit time. Then, if $k$ is the factor of proportionality (a hardness constant),

$$QQ_2 = k(f)(p)(n)(2\pi y).$$

If $QQ_1$ is to be constant for all points $Q$, then

$$QQ_1 = (QQ_2) \sec\theta = (kfpn)(y \sec\theta) = \text{constant},$$

where $\theta$ is the angle between the normal and the axis. In short, $y \sec\theta = a$, a constant. Thus, if $QF$ be drawn tangent to the curve, $QF=a$, a definitive property of the tractrix. This is the form of the *Schiele* pivot mentioned in some books on Mechanics.

We end the account with a list of rectangular equations of the tractrix:

$$x = a \sinh^{-1} \sqrt{a^2y^{-2} - 1} - \sqrt{a^2 - y^2}$$

$$x = a \operatorname{sech}^{-1} \frac{y}{a} - \sqrt{a^2 - y^2}$$

$$x = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}} - \sqrt{a^2 - y^2}$$

$$x = t - a \tanh \frac{t}{a}, \qquad y = a \operatorname{sech} \frac{t}{a} \cdot$$

They seem relatively useless.

## MATHEMATICAL EDUCATIONAL NOTES

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### GEOMETRY IN THE FIRST GRADE

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In the spring of 1958 we spent two and a half months in an experiment which involved teaching geometrical notions and constructions to the entire class of first grade students at Stanford Elementary School (a public school in the Palo Alto Unified School District). After a few informal talks we began a systematic development of a modified version of Book I of Euclid's *Elements*.

Our approach was to stimulate reasoning among the pupils, although no formal proofs were attempted. The propositions from Euclid which were stressed were the constructions. In each case the construction (*e.g.*, bisecting a line segment) was presented as an open problem, and the students were encouraged to attempt solutions. As much as possible we forced the children to give the reasons for rejecting an incorrect solution. Generally speaking, our pedagogical procedure closely resembled that of Socrates' interrogation of the slave in Plato's dialogue *Meno*. It is worth noting that at no point did we rely on any knowledge of arithmetic. As a consequence our program is completely independent of the standard curriculum in elementary school mathematics.

Our main modification of Euclid was to use the compasses as rigid instruments to make direct comparisons of distances. We thereby trivialized Propositions 2 and 3 of Book I.